MAB241 COMPLEX VARIABLES

BILINEAR MAPPING

1 The transformation

A transformation $T(z) : z \to w$ is called bilinear if it takes the form

$$w = T(z) = \frac{az + b}{cz + d}. \tag{1}$$

You should have already had practice with finding a bilinear transformation from 3 point mappings. In this handout we will cover the image of a set under a bilinear transformation.

2 Image of a set

Here we consider how to find the image of a given set S which may be either a circle or a straight line. An important property of bilinear maps is that both a straight line or a circle will always map either a straight line or a circle. So by recognising which of the two images the map produces, we can use the right calculations to find the equation of the circle or straight line.

As with any mapping, it is useful to sketch out two complex planes, one for S and one for the images. Drawing out the given set on the domain will help prevent simple mistakes with algebra. Every bilinear transformation has exactly one singularity, at the point $z = -\frac{d}{c}$. This point is mapped to infinity, so if the line or circle you are mapping contains this singular point, you know that the image contains the point at infinity, if S doesn't contain the singular point, then it maps onto a circle.

2.1 Example

Given the bilinear transformation

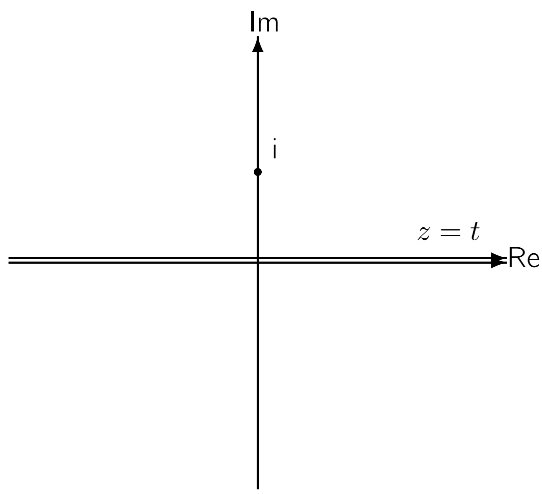
$$w = T(z) = \frac{z + \mathrm{i}}{z - \mathrm{i}}, \tag{2}$$

find the image of the line $z = t,\ t \in \mathbb{R}$.

<u>Solution</u>

We can draw this line on an Argand diagram. Plot and label the singularity on the diagram, here it is $z =$i. The singularity does not lie on the line $z = t$ so its image must be a circle.

Now we know this we try to manipulate the function into the circle equation, we start by substituting the equation of the line into the transformation.



There are many ways to find the equation of a circle, here we will split the function into real and imaginary parts, so finding u and v in $w = T(z) = u + \mathrm{i}v$. We start this by multiplying multiplying the numerator and denominator by the complex conjugate of the denominator.

$$T(z) = \frac{t+\mathrm{i}}{t-\mathrm{i}} = \frac{(t+\mathrm{i})(t+\mathrm{i})}{(t-\mathrm{i})(t+\mathrm{i})} = \frac{t^2-1+2t\mathrm{i}}{t^2+1}.$$

We can then split this into real and imaginary parts to find

$$u = \frac{t^2-1}{t^2+1} = 1 - \frac{2}{t^2+1}, \quad v = \frac{2t}{t^2+1}.$$

Now we must solve for t, in this case, we can substitute v into u

$$u = 1 - \frac{v}{t}.$$

Rearranging this gives the equation solved for t;

$$t = \frac{-v}{u-1}. \tag{3}$$

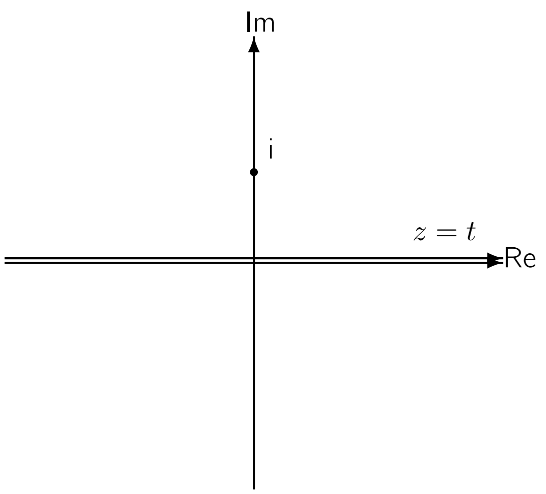
We can now substitute (3) back into either of the u or v equation. We will do it from the u equation. Hence

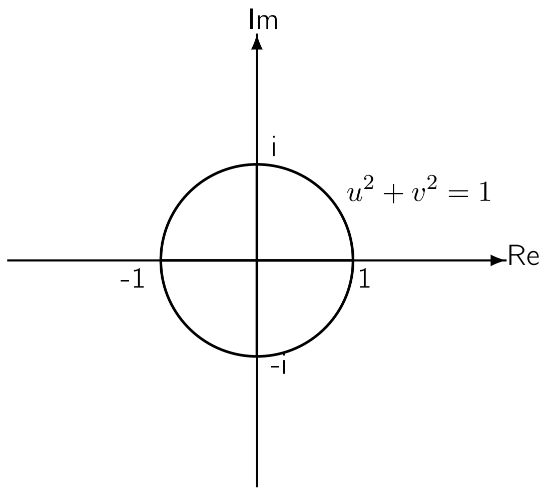
$$u = 1 - \frac{2}{\left(\frac{v}{1-u}\right)^2 + 1} \qquad \Rightarrow \quad (u-1)\left(\left(\frac{-v}{u-1}\right)^2 + 1\right) = -2$$

$$\Rightarrow \frac{v^2}{u-1} + (u-1) = -2 \qquad \Rightarrow \quad v^2 + (u-1)^2 = -2(u-1)$$

$$\Rightarrow v^2 + u^2 - 2u + 1 = -2u + 2 \quad \Rightarrow \quad v^2 + u^2 = 1.$$

Therefore we have found that this function maps the line $z = t,\ t \in \mathbb{R}$ onto the unit circle centred on the origin.





2.2 Example

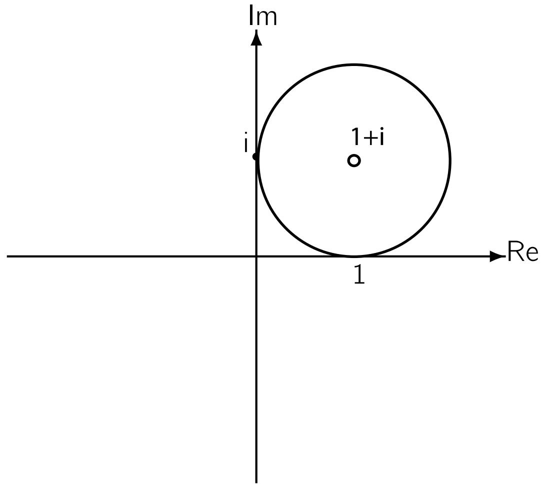
For the same bilinear transformation as example 2.1, find the image of the circle $\{z : |z - 1 - \mathrm{i}| = 1\}$

Solution

$$T(z) = \frac{t + \mathrm{i}}{t - \mathrm{i}}$$

As in example 2.1, we start by plotting the circle and the singular point on the Argand diagram. From this we can see that the singularity lies on the circle. This means that it maps to the point at infinity, and therefore the image is a straight line.

To find the equation of a straight line, we only need two points on the line. To find these points we take the transformation of two points on the domain different to the singular point. When choosing which points to transform, use points which are easy to calculate with.

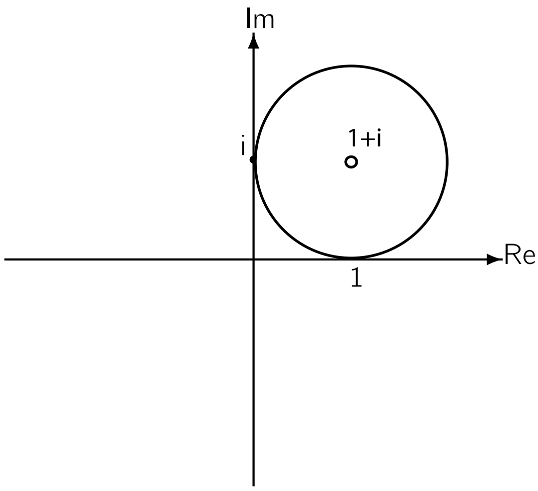


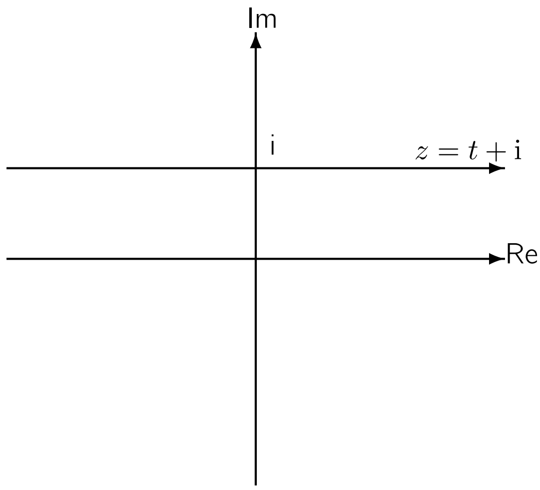
We take any two points on the circle which are not $z = i$, and use them to find 2 points which are on the image line;

$$\text{at } z = 2 + \mathrm{i},\ T(2 + \mathrm{i}) = \frac{2 + 2\mathrm{i}}{2} = 1 + \mathrm{i}, \tag{4}$$

$$\text{at } z = 1,\ T(1) = \frac{1 + \mathrm{i}}{1 - \mathrm{i}} = \frac{2\mathrm{i}}{2} = \mathrm{i}. \tag{5}$$

So the image line, which goes through these two points, can be written as $w = t + \mathrm{i}$ with $t \in \mathbb{R}$.





3 Key Points

- Sketch out two Argand diagrams, one for the given set S, and one for the image set.
- Draw the line or circle that is to be mapped, and label the singular point on the mapping.
- If the singular point of T lies on S it means that tthe point is mapped to infinity under the transformation, and therefore the image must be a straight line stretching to infinity in both directions. Find the mapping of two other points z_1, z_2 and the image will be the line that connects the points $T(z_1)$ and $T(z_2)$.
- If the singular point does not lie on S, the image is always a circle. Find the expressions for $w = u + \mathrm{i}v$ in terms of z and then solve for z. Substitute this back into either the u or v equations and manipulate to get an expression of form.

  $$(u - u_1)^2 + (v - v_1)^2 = R^2$$

  Which is a circle centred on (u_1, v_1) with radius R.
- These kind of questions require a lot of calculation. However, if you learn the theory knowing what to do for the calculations will become a lot clearer.

For more information on bilinear mappings refer to the lecture notes.