COMPLEX VARIABLES

RESIDUE THEOREM

1 The residue theorem

Suppose that the function f is analytic within and on a positively oriented simple closed contour C except for a finite number of isolated singular points $\{z_j, j = 1, 2, ..., N\}$ *interior to C, then*

$$\int_C f(z)\mathrm{d}z = 2\pi\mathrm{i}\sum_{j=1}^{N} \operatorname*{Res}_{z=z_j} f(z). \tag{1}$$

A proof of this can be found in the lecture notes.
This is a very important result and can help us calculate integrals around contours that would be impossible to do using standard single variable calculus. The residue theorem can even be used when integrating along the real line.

2 Integrals around closed curves

The most obvious way of using this theorem is for finding an integral around a simple closed contour enclosing a finite number of singularities.

2.1 Example

Evaluate

$$I = \int_{C_1(1)} \frac{z}{z^2 - 1}\,\mathrm{d}z. \tag{2}$$

<u>Solution</u>
By factorizing the denominator of the integrand we get

$$\frac{z}{z^2 - 1} = \frac{z}{(z-1)(z+1)}.$$

Here we can see that the two poles of this function are at $z = \pm 1$, note that both these poles are simple. Only one of these poles, $z = 1$, is inside the contour, so we need to calculate the residue at this pole

$$\operatorname{Res} f(1) = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}(z-1)\frac{z}{(z-1)(z+1)} = \lim_{z\to 1}\frac{z}{(z+1)} = \frac{1}{2}.$$

Now using the residue theorem we evaluate I by multiplying the sum of the residues by $2\pi\mathrm{i}$ to get

$$I = \int_{C_1(1)} \frac{z}{z^2 - 1}\,\mathrm{d}z = 2\pi\mathrm{i}\frac{1}{2} = \pi\mathrm{i}.$$

Compare this result to example 2.1 of the Cauchy integral formula handout. You will notice that this theorem is just an extension of the formula.

3 Integrals along the real line

This theorem also has applications when integrating along the real line. Some real integrals cannot be evaluated by normal calculus, this is because the integrand does not have a "simple" anti-derivative (see the fundamental theorem of calculus). However we can evaluate them using complex variables and the residue theorem. This is one of the most important applications of the theory of residues.

3.1 Some notation

$C_R^+(0)$ - a semi circle of radius R centred on the origin in the top half of the complex plane, not including the diameter.
$\{-R, R\}$ - The line on the real axis, between but not including, $-R$ and R, also the diameter of C_R^+.
C^+ - The closed contour made of $\lim_{R\to\infty}\left[C_R^+(0)\cup\{-R,R\}\right]$.

$$I_{C^+} = \int_{C^+} f(z)\,\mathrm{d}z, \quad I_{C_\infty^+(0)} = \lim_{R\to\infty}\int_{C_R^+(0)} f(z)\,\mathrm{d}z, \quad I = \lim_{R\to\infty}\int_{\{-R,R\}} f(z)\,\mathrm{d}z. \tag{3}$$

It follows that

$$I = \left(I_{C^+}\right) - \left(I_{C_\infty^+(0)}\right). \tag{4}$$

3.2 3 step process

The whole process of calculating integrals using residues can be confusing, and some text books show the method in a slightly different way to the lecture notes. Here we have split the process down to 3 steps, so you can follow what you are doing and not miss out important results.

- Step 1 is preliminaries, this involves assigning the real function in the original integral to a complex function on the plane, and also identifying the singularities of this function.
- Step 2 is checking that the integral along the contour $C_R^+(0)$ converges to 0 as $R\to\infty$. This will mean that the integral along the real line $(-R, R)$ as $R\to\infty$ is equal to the integral along the closed contour I_{C^+} minus $\lim_{R\to\infty} I_{C_R^+(0)}$.
- Step 3 is using the residue theorem to evaluate the integral I_{C^+} by calculating the residues at the singularities found in step one that lie above the real axis.

3.3 Example

Evaluate

$$I = \int_{-\infty}^{\infty}\frac{\mathrm{d}x}{x^2+1}.$$

Solution

Step 1

We define the complex integral

$$I_{C^+} = \int_{C^+} f(z)\mathrm{d}z, \ , \qquad f(z) = \frac{1}{z^2+1}.$$

Identify the singularities

$$(z^2+1) = (z-z_1)(z-z_2) \ \text{ with } \ z_1 = \mathrm{i}, \ z_2 = -\mathrm{i}.$$

Step 2
On $C_R^+(0)$, $|z| = R$, so we can use the triangle inequality to get

$$\left|z^2+1\right| = \left|z-z_1\right|\left|z-z_2\right| \geq \Big|\left|z\right| - \left|z_1\right|\Big|\Big|\left|z\right| - \left|z_2\right|\Big| = (R-1)^2.$$

Thus

$$f(z) = \frac{1}{z^2+1} \leq \frac{1}{(R-1)^2}.$$

Using the bounding an integral theorem we can show

$$\left|\int_{C_R^+} f(z)\,\mathrm{d}z\right| \leq \frac{1}{(R-1)^2}\pi R.$$

Note that as $R \to \infty$, $\frac{1}{(R-1)^2}\pi R \to 0$ so by squeeze theorem so the integral must also tend to 0.

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2+1} = \int_{C^+} \frac{1}{z^2+1}\mathrm{d}z.$$

Step 3
Find the residue at $z = \mathrm{i}$

$$\operatorname*{Res}_{z=\mathrm{i}} f(z) = \lim_{z\to\mathrm{i}}(z-\mathrm{i})\frac{1}{z^2+1} = \lim_{z\to\mathrm{i}}\frac{(z-\mathrm{i})}{(z-\mathrm{i})(z+\mathrm{i})} = \lim_{z\to\mathrm{i}}\frac{1}{z+\mathrm{i}} = \frac{1}{2\mathrm{i}} = -\frac{1}{2}\mathrm{i}.$$

Therefore, by residue theorem

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2+1} = \int_{C^+} \frac{1}{z^2+1}\mathrm{d}z = 2\pi\mathrm{i}(-\frac{\mathrm{i}}{2}) = \pi.$$

And so our final answer is

$$I = \pi.$$

This result could also have been evaluated by seeing that $\frac{1}{x^2+1}$ is just the derivative of $\arctan(x)$.

3.4 Example

Evaluate

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x^2+2x+2}\,\mathrm{d}x \tag{5}$$

Solution
This cannot be calculated using any techniques learnt in first year (seriously, you can try it).
Step 1
We can recognise that $\sin x$ is simply the imaginary part of $\mathrm{e}^{\mathrm{i}x}$

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x^2+2x+2}\,\mathrm{d}x = \operatorname{Im}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x^2+2x+2}\,\mathrm{d}x$$

Now we consider

$$I_{C^+} = \int_{C^+} f(z)\mathrm{d}z, \;, \qquad f(z) = \frac{\mathrm{e}^{\mathrm{i}z}}{z^2+2z+2}.$$

Identify the singularities

$$(z^2+2z+2) = (z-z_1)(z-z_2) \text{ with } z_1 = -1+\mathrm{i},\; z_2 = -1-\mathrm{i}.$$

$$|-1+\mathrm{i}| = |-1-\mathrm{i}| = \sqrt{2}$$

Step 2

We also note that on $C_R^+(0)$, the imaginary part of z, is always greater than 0, so

$$|\,\mathrm{e}^{\mathrm{i}z}\,| = |\,\mathrm{e}^{-y+\mathrm{i}x}\,| = |\,\mathrm{e}^{-y}\,||\,\mathrm{e}^{\mathrm{i}x}\,| = |\,\mathrm{e}^{-y}\,| \leq 1,$$

and

$$\left|z^2+2z+2\right| = \left|z-z_1\right|\left|z-z_2\right| \geq \Big||z|-|z_1|\Big|\Big||z|-|z_2|\Big| = (R-\sqrt{2})^2.$$

Thus

$$\Big|f(z)\Big| = \Big|\frac{\mathrm{e}^{\mathrm{i}z}}{z^2+2z+2}\Big| \leq \frac{1}{(R-\sqrt{2})^2}.$$

Thus, by seeing that a length of $C_R^+(0)$ is πR and using the theorem for bounding an integral, we get

$$\left|\int_{C_R^+(0)} f(z)\mathrm{d}z\right| \leq \frac{1}{(R-\sqrt{2})^2}\pi R.$$

Note that as $R \to \infty$, $\frac{1}{(R-\sqrt{2})^2}\pi R \to 0$ so by squeeze theorem so the integral must also tend to 0.

Step 3

Now we know to use the residue theorem to evaluate the integral as $R \to \infty$. To do this we find the sum of all the residues inside C^+, and multiply it by $2\pi\mathrm{i}$.

Only z_1 is inside C^+, thus

$$\begin{aligned}
I_{C^+} &= 2\pi\mathrm{i} \lim_{z\to z_1} (z-z_1)f(z) \\
&= 2\pi\mathrm{i} \lim_{z\to z_1} (z-z_1)\frac{\mathrm{e}^{\mathrm{i}z}}{(z-z_1)(z-z_2)} \\
&= 2\pi\mathrm{i}\frac{\mathrm{e}^{\mathrm{i}z_1}}{z_1-z_2} = 2\pi\mathrm{i}\frac{\mathrm{e}^{\mathrm{i}(-1+\mathrm{i})}}{2\mathrm{i}} = \frac{\pi}{\mathrm{e}}\,\mathrm{e}^{-\mathrm{i}}.
\end{aligned}$$

This is the value of the integral of the top half of the plane, in the first step of this solution we let the real integral I be equal the Imaginary part of the complex integral I_{C^+}, so the integral along the real line is

$$I = \mathrm{Im}\ I_{C^+} = \frac{\pi}{\mathrm{e}}\sin 1.$$

4 Key points

- The residue theorem is combines results from many theorems you have already seen in this module, try using it with previous examples in problem sheets that you would have used Cauchy's Theorem and Cauchy's integral formula on.
- When calculating integrals along the real line, Argand diagrams are a good way of keeping track of which contours you are integrating, and where the singularities lie.
- This 3 step method is just a guide to help remember the steps of the process, when revising this kind of question you may wish to think of your own steps or way of remembering the process.

For more information on residue theorem refer to the lecture notes.