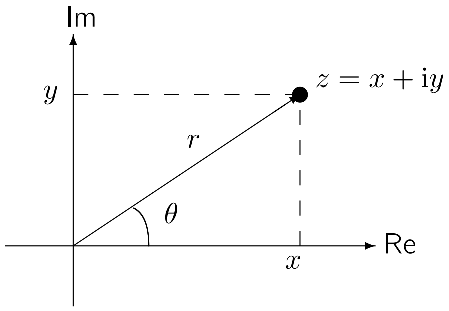
MAB241 COMPLEX VARIABLES

MODULUS AND ARGUMENT

1 Modulus and argument

- A complex number is written in the form

$$z = x + \mathrm{i}y.$$



- The **modulus** of z is

$$|z| = r = \sqrt{x^2 + y^2}.$$

- The **argument** of z is

$$\arg z = \theta = \arctan\left(\frac{y}{x}\right).$$

  Note: When calculating θ you must take account of the quadrant in which z lies - if in doubt draw an Argand diagram.

- The **principle value of the argument** is denoted by $\operatorname{Arg} z$, and is the unique value of $\arg z$ such that $-\pi < \arg z \le \pi$. $\operatorname{Arg} z$ in obtained by adding or subtracting integer multiples of 2π from $\arg z$.

- Writing a complex number in terms of **polar coordinates** r and θ:

$$z = x + \mathrm{i}y = r\cos\theta + \mathrm{i}r\sin\theta = r(\cos\theta + \mathrm{i}\sin\theta) = r\,\mathrm{e}^{\mathrm{i}\theta}.$$

- For any two complex numbers z_1 and z_2

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \quad \text{and, for } z_2 \ne 0, \quad \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 + \arg z_2.$$

2 Some examples

2.1 Example

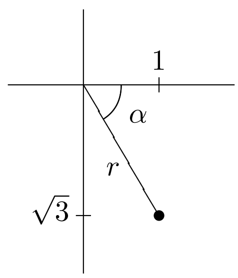
Calculate $\arg z$ and $\operatorname{Arg} z$ where

$$z = 1 - \sqrt{3}\,\mathrm{i}.$$

<u>Solution</u>

Let $\alpha = \arctan\left|\frac{y}{x}\right|$, and use the Argand diagram to determine θ.
So

$$\alpha = \arctan\left|\frac{-\sqrt{3}}{1}\right| = \pi/3,$$



where the acute angle is chosen for α.
Here z lies in the 4^{th} quadrant, therefore for any integer n

$$\arg z = \theta = 2n\pi - \pi/3,$$

but

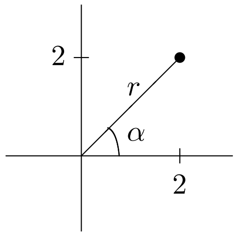
$$\operatorname{Arg} z = -\pi/3.$$

2.2 Example

Write $z = 2 + 2\mathrm{i}$ in the form $r\,\mathrm{e}^{\mathrm{i}\theta}$, where $r, \theta \in \mathbb{R}$ with $r > 0$ and $\theta \in (-\pi, \pi]$.

<u>Solution</u>

Here $x = 2$ and $y = 2$, so that $r = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$ and



$$\alpha = \arctan\left|\frac{2}{2}\right| = \pi/4.$$

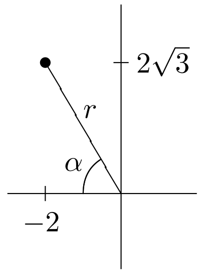
Here z lies in the 1st quadrant, therefore $\arg z = \theta = \pi/4$; hence $z = 2\sqrt{2}\,\mathrm{e}^{\mathrm{i}\pi/4}$.

2.3 Example

Write $z = -2 + 2\sqrt{3}\,\mathrm{i}$ in the form $r\,\mathrm{e}^{\mathrm{i}\theta}$, where $r, \theta \in \mathbb{R}$ with $r > 0$ and $\theta \in (-\pi, \pi]$.

<u>Solution</u>

Here $x = -2$ and $y = 2\sqrt{3}$, so that $r = \sqrt{(-2)^2 + (2\sqrt{3})^2} = 4$ and



$$\alpha = \arctan\left|\frac{2\sqrt{3}}{-2}\right| = \pi/3.$$

Here z lies in the 2nd quadrant, therefore $\arg z = \theta = \pi - \pi/3 = 2\pi/3$: hence $z = 4\,\mathrm{e}^{2\mathrm{i}\pi/3}$.

2.4 Example

Let $\arg(z_1) = 5\pi/6$ and $\arg(z_2) = -\pi/3$, calculate (a) $\mathrm{Arg}(z_1 z_2)$. (b) $\mathrm{Arg}\left(\dfrac{z_1}{z_2}\right)$.

<u>Solutions</u>

(a) $\arg(z_1 z_2) = \arg z_1 + \arg z_2 = 5\pi/6 + (-\pi/3) = \pi/2 \Rightarrow \mathrm{Arg}(z_1 z_2) = \pi/2$.

(b) $\arg(z_1/z_2) = \arg z_1 - \arg z_2 = 5\pi/6 - (-\pi/3) = 7\pi/6 \Rightarrow \mathrm{Arg}(z_1/z_2) = 7\pi/6 - 2\pi = -5\pi/6$.

For more information on calculating the modulus and argument of complex numbers refer to the lecture notes.