SYMBΩL

COMPLEX VARIABLES

ANALYTIC FUNCTIONS

1 Cauchy-Riemann equations

Showing that a function is analytic within an open region is a lot simpler than it first appears. The definition of analyticity requires that every point within the region the function is differentiable. Using the Cauchy-Riemann equations we only have to find first partial derivatives to get the terms we need to show the function is analytic.

Theorem

The function $f(z) = u(x, y) + \mathrm{i}v(x, y)$, where u and v are real-valued, is analytic in a domain D if and only if at every point in D the first partial derivatives of u and v exist and are continuous and satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1}$$

Since this theorem is an "if and only if" statement we can use this set of equations to prove whether or not a function is analytic. If the equations are satisfied for a region, its analytic, if the equations are not satisfied in a region, the function is not analytic.

2 How to show a function is analytic

2.1 Example

Let $f(z) = e^{\mathrm{i}z}$, show that $f(z)$ is entire (analytic everywhere).

Solution

Firstly we need to get the function into the form$f(z) = u(x, y) + \mathrm{i}v(x, y)$. We do this using the definition of the exponential and Eulers equation.

$$f(z) = e^{\mathrm{i}z^2} = e^{\mathrm{i}(x^2-y^2+\mathrm{i}xy)} = e^{\mathrm{i}(x^2-y^2)-xy} = e^{-xy}e^{\mathrm{i}(x^2-y^2)},$$

$$f(z) = e^{-xy}(\cos(x^2-y^2) + \mathrm{i}\sin(x^2-y^2)) = e^{-xy}\cos(x^2-y^2) + \mathrm{i}e^{-xy}\sin(x^2-y^2).$$

So now we have split the function into real and imaginary part, we get the function into the form

$$u(x, y) = e^{-xy}\cos(x^2-y^2),$$
$$v(x, y) = e^{-xy}\sin(x^2-y^2).$$

Now we use partial differentiation to get

$$\frac{\partial u}{\partial x} = -e^{-xy}\sin(x^2-y^2) \qquad \frac{\partial v}{\partial y} = -e^{-xy}\sin(x^2-y^2), \quad \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}.$$
$$\frac{\partial u}{\partial y} = -e^{-xy}\cos(x^2-y^2) \qquad \frac{\partial v}{\partial x} = e^{-xy}\cos(x^2-y^2), \quad \Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

So the function is analytic whenever these equations are satisfied and continuous, which is for all x and for all y. So the function is entire.

3 Key points

- Always split the function into real and imaginary parts, identify these as functions $u(x, y)$ and $v(x, y)$ respectively.
- Use partial differentiation to find the terms in the Cauchy-Riemann equations, the function is analytic only where the equations are satisfied and continious.

For more information on analytic functions refer to section 1.7 in the lecture notes.