MAB241 COMPLEX VARIABLES

PARAMETERISATION OF CURVES IN THE COMPLEX PLANE

1 Contours

When integrating a complex function $f(z)$ along a contour C between two points z_1 and z_2 in the complex plane, we can't always use simple intervals as we do in single variable calculus . To perform the integration we describe the contour as some function $z = z(t)$ on an interval $a \leq t \leq b$, where $z(a) = z_1$ and $z(b) = z_2$. Substituting this into the integral we get

$$\int_C f(z)\,\mathrm{d}z = \int_a^b f(z(t))\frac{\mathrm{d}z}{\mathrm{d}t}\,\mathrm{d}t \tag{1}$$

which can be calculated using standard methods.

2 Straight line

When parameterising a straight line it is useful to draw the line on an Argand diagram to help . Then we need to find the parametric function $z(t)$. A general form we can use for straight lines is

$$z(t) = (1-t)z_1 + tz_2 \text{ with } 0 \leq t \leq 1. \tag{2}$$

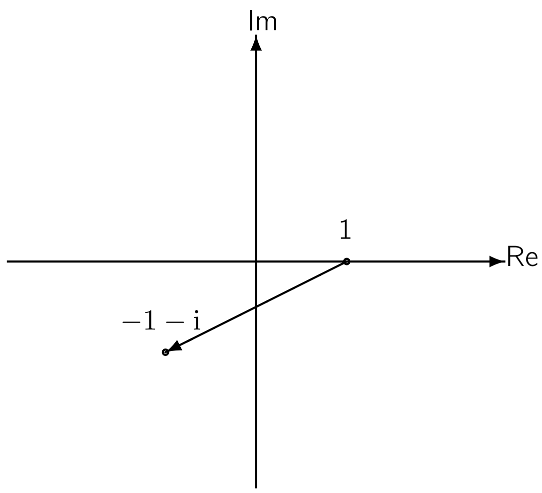
This works for any straight line because when $t = 0, z(t) = z_1$ and, as t increases from 0 to 1, $z(t)$ follows a linear path until it reaches z_2.

2.1 Example

Let C be the line from $z_1 = 1$ to $z_2 = -1 - \mathrm{i}$.
Calculate

$$I = \int_C |z|^2\,\mathrm{d}z. \tag{3}$$

<u>Solution</u>
With any problem of this kind it is always useful to draw out an Argand diagram, label the start and end points and draw the contour connecting these points, take into account the direction of the contour.



The contour in the diagram is the line on which we are integrating, and it's this line that we must parameterise in terms of t.

Firstly substitute the the endpoints z_1 and z_2 into equation (2) to get the parameterisation. This gives

$$z(t) = (1-t)1 + t(-1-\mathrm{i}) = 1 - 2t - \mathrm{i}t \qquad \text{with} \quad 0 \leq t \leq 1.$$

Substitute $z(t)$ into the integrand. Note: $\frac{dz}{dt} = (-2-\mathrm{i})$ so multiply the integrand by $(-2-\mathrm{i})$ to get

$$I = \int_0^1 |1-2t-\mathrm{i}t|^2(-2-\mathrm{i})\,\mathrm{d}t.$$

$(-2-\mathrm{i})$ is a constant and can be taking out of the integral, then further simplification yields

$$I = (-2-\mathrm{i})\int_0^1 ((1-2t)^2+t^2)\,\mathrm{d}t = -(2+\mathrm{i})\int_0^1 (1-4t+5t^2)\,\mathrm{d}t.$$

Now you can integrate the function to obtain

$$I = (-2-\mathrm{i})\Big[t-2t^2+\frac{5}{3}t^3\Big]_0^1 = -\frac{4}{3}-\frac{2\mathrm{i}}{3},$$

and so the final result we get is

$$I = \int_C |z|^2\,\mathrm{d}t = -\frac{4}{3}-\frac{2\mathrm{i}}{3}.$$

3 Arcs

When integrating along a full circle, or an arc of a circle, in the anti-clockwise direction, we use the following parameterisation

$$z(\theta) = z_0 + R\,\mathrm{e}^{\mathrm{i}\theta} \quad \text{with } \alpha \le \theta \le \beta. \tag{4}$$

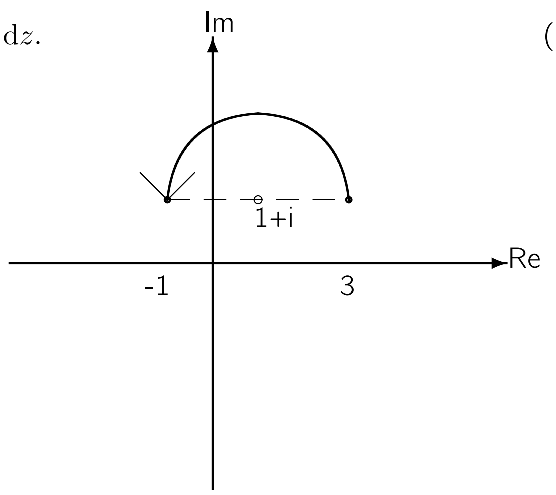
Here z_0 is the centre of the circle, R is the radius of the circle, and α and β are the angles from z_0 at z_1 and z_2. So for a full circle you would use $\theta \in [0, 2\pi)$. Another important thing to consider is the direction of the contour, as θ increases in the above equation the contour moves round anti-clockwise, to make it move in the clockwise direction use

$$z(\theta) = z_0 + R\,\mathrm{e}^{-\mathrm{i}\theta} \text{ with } \alpha \le \theta \le \beta. \tag{5}$$

3.1 Example

Let $C_2^+(z_0)$ be the top half of a circle of radius 2 centred on point $z_0 = (1+\mathrm{i})$ oriented in the anti-clockwise direction, evaluate

$$I = \int_{C_2^+(z_0)} (\mathrm{Re}\,z)\,\mathrm{d}z. \tag{6}$$



Solution

Once again we draw out the Argand diagram so we can visualise what it is we're calculating.

The contour in the diagram is the line along which we are integrating, and it's this line that we must parameterise in terms of θ.

Since we are going in the anti-clockwise direction we use parameterisation in equation (4) to get

$$z(\theta) = (1+\mathrm{i}) + 2\,\mathrm{e}^{\mathrm{i}\theta} \text{ with } 0 \le \theta \le \pi.$$

Substitute $z(\theta)$ into the integrand. Note: $\frac{dz}{d\theta} = 2\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}$ so multiply the integrand by $2\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}$ to get

$$I = \int_0^\pi \mathrm{Re}\,(2\,\mathrm{e}^{\mathrm{i}\theta} + 1 + \mathrm{i})2\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}\;\mathrm{d}\theta.$$

Simplify the integrand to get

$$I = \int_0^{\pi} (2\cos\theta + 1)2\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta = \int_0^{\pi} 4\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}\cos\theta + 2\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}\ \mathrm{d}\theta.$$

The trick here is to split the $4\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}\cos\theta$ term into its complex parts, so that

$$4\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}\cos\theta = 2\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}(\mathrm{e}^{-\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\theta}) = 2\mathrm{i} + 2\mathrm{i}\,\mathrm{e}^{2\mathrm{i}\theta}.$$

Putting this back into the integral we get

$$I = \int_0^{\pi} 2\mathrm{i} + 2\mathrm{i}\,\mathrm{e}^{2\mathrm{i}\theta} + 2\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}\ \mathrm{d}\theta.$$

We can integrate this to get

$$I = \Big[2\mathrm{i}\theta + \mathrm{e}^{2\mathrm{i}\theta} + 2\,\mathrm{e}^{\mathrm{i}\theta}\Big]_0^{\pi} = -4 + 2\pi\mathrm{i}.$$

So the final answer is

$$\int_{C_2^+(z_0)} (\mathrm{Re}\,z)\mathrm{d}z = -4 + 2\mathrm{i}\pi.$$

4 Key Points

- Sketch out the Complex plane, label endpoints and draw the contour, include a directional arrow.
- Find a parameterisation for the contour using one of the above methods or your own intuition. Make sure the endpoints satisfy $z(a) = z_1$ and $z(b) = z_2$.
- Substitute the parameterisation into the integral, remember now you're integrating with respect to t (or θ), so multiply the integrand by $\frac{dz}{dt}$ (or $\frac{dz}{d\theta}$).
- Once you have done this you can simplify and integrate as normal.

For more information on parameterisation of curves refer to the lecture notes.